EFFECTS OF HIGH SUBSONIC FLOW ON SOUND PROPAGATION IN A VARIABLE-AREA DUCT*

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SUMMARY

The propagation of sound in a converging-diverging duct containing a quasi-one-dimensional steady flow with a high subsonic throat Mach number is studied. The behavior of linearized acoustic theory at the throat of the duct is shown to be singular. This singularity implies that linearized acoustic theory is invalid. The explicit singular behavior is determined and is used to sketch the development (by the method of matched asymptotic expansions) of a non-linear theory for sound propagation in a sonic throat region.

1. INTRODUCTION

Observations of a correlation between axial Mach number and attenuation of sound radiated upstream from so-called sonic engine inlets have recently focused attention on the acoustic behavior of variable-geometry ducts (refs. 1 and 2). For high-subsonic flows in these ducts, non-linear transonic effects become of major interest. In the linear case, a fully three-dimensional theory presents formidable computational difficulties, and a study of possible non-linear effects is, of course, even more complicated. Thus, it is natural, in undertaking such an effort, to restrict attention initially to a quasi-one dimensional model: the simplest case likely to lead to results of some practical interest. Many earlier authors have studied linear quasi-one dimensional duct acoustics (see refs. 2-6, for example), but, in general, these studies have not been concerned with either the behavior or the validity of the linerized solution as the axial Mach number approaches unity.

The present paper presents some results of an ongoing analytical study of quasi-one dimensional acoustics in converging-diverging ducts with high-subsonic throat Mach numbers. The problem is inherently nonlinear, much like steady

*This work was supported by NASA Langley Research Center through the Acoustics Branch, ANRD.

transonic flow theory, but the nonlinear behavior occurs only in a narrow region surrounding the throat section. Linearized theory yields a singular solution in this region, and the current study is employing the method of matched asymptotic expansions to determine the proper solution. However, in order to apply any asymptotic method to correct a solution which is not uniformly valid, it is necessary to know in detail the singular behavior of the defective solution.

Thus, it is the major purpose of this paper to study the nature of the singularity of linearized theory at a sonic throat. To correct the defect using asymptotic methods requires an intricate analysis; the problem depends crucially on two small parameters, and the nonlinear correction to the defect in linearized theory involves a distinguished limit for small values of these parameters. The results obtained here are necessary preliminaries in this analysis. However, they also are of independent interest and do not appear to have been discussed previously.

The analysis presented in sections 3 and 4 naturally suggests that linearization is inappropriate in a small region near the throat of the duct. The detailed results concerning the singular behavior of the linear solution lead to an appropriate stretching of the space variable and a corresponding inner expansion of the dependent variables which does not suffer a singularity at the throat. In the final section of this work the equations describing this inner nonlinear theory are presented, although the details of the expansion process are omitted for brevity. The solution of these nonlinear equations is the subject of current research and will appear in subsequent publications.

2. FORMULATION AND ACOUSTIC PERTURBATION

We consider the propagation of sound in a variable area duct carrying a homentropic inviscid ideal gas flow. The acoustic wavelength is assumed sufficently large, and the area variation sufficiently slow, that the field can be described by the equations of quasi-one dimensional gas dynamics (ref. 7):

$$
\begin{aligned}
c_s\bar{\rho}_t + \bar{u}\bar{\rho}_x + \bar{\rho}\bar{u}_x + \bar{\rho}\bar{u}(A'/A) &= 0 \\
c_s\bar{u}_t + \bar{u}\bar{u}_x + (1/\bar{\rho})\,\bar{p}_x &= 0 \qquad (2.1) \\
\bar{p}/\bar{\rho}^{\gamma} = \text{constant} &= B
\end{aligned}
$$

In equations (2.1) $\bar{p}$, $\bar{\rho}$, and $\bar{u}$ are the total fluid pressure, density, and axial velocity, and $A(x)$ is the duct cross sectional area. The dimensionless independent variables x and t are measured in units of L and L/c_s respectively, where L is a characteristic length associated with the area variation, and c_s is the stagnation value of sound speed in the gas. The geometry of the problem is as indicated in figure 1 where the origin of x corresponds to a throat: $A'(0)=0$.

If the velocity and density in the basic steady flow in the duct are denoted by $U(x)$ and $R(x)$ respectively, then from (2.1),

$$UR' + RU' + RU(A'/A) = 0 \quad , \quad \text{and} \quad UU' + \gamma BR^{\gamma-2}R' = 0 \tag{2.2}$$

where the energy relation in equations (2.1) has been used to eliminate the pressure from the system. We intend to seek solutions to the system (2.1) which are small perturbations about the steady values U and R, and it is convenient at the outset to define dimensionless variables $u(x,t)$ and $\rho(x,t)$ according to

$$\bar{u}(x,t) = U(x)[1+u(x,t)] \quad , \quad \text{and} \quad \bar{\rho}(x,t) = R(x)[1+\rho(x,t)] \tag{2.3}$$

Substituting (2.3) into (2.1) and employing the steady relations (2.2) yields the system of equations on u and ρ in the form

$$\begin{aligned} &G^{\frac{1}{2}}\rho_t + M[(1+u)\rho_x+(1+\rho)u_x] = 0 \\ &G^{\frac{1}{2}}u_t + M(1+u)u_x+(1/M)(1+\rho)^{\gamma-2}\rho_x+(M'/G)[(1+u)^2-(1+\rho)^{\gamma-1}] = 0 \end{aligned} \tag{2.4}$$

In equations (2.4), $M(x)$ is the flow Mach number $U(x)/c(x)$, $c(x)$ is the speed of sound in the steady flow ($c^2=\gamma BR^{\gamma-1}$), and

$$G(x) = (c_s/c)^2 = 1 + (\gamma-1)M^2/2 \tag{2.5}$$

the latter expression following from the Bernoulli relation implied by the second of equations (2.2). Equations (2.4) are equivalent to those used by Cheng and Crocco in reference 3.

We introduce a small dimensionless parameter δ, which measures the strength of the source of sound in the duct, and is assumed given from the boundary conditions associated with the system (2.1). Then $u(x,t)=u(x,t;\delta)$, $\rho(x,t)=(x,t;\delta)$ which we assume to have expansions for $\delta<<1$ of the form

$$u = \delta\mu(x,t) + \ldots \quad , \quad \rho = \delta r(x,t) + \ldots \tag{2.6}$$

Substituting (2.6) into (2.4) and neglecting all but first-order terms we obtain the linearized acoustic equations:

$$\begin{aligned} &G^{\frac{1}{2}}r_t + M(r_x+\mu_x) = 0 \\ &G^{\frac{1}{2}}\mu_t + M\mu_x + (1/M)r_x+(M'/G)[2\mu-(\gamma-1)r] = 0 \end{aligned} \tag{2.7}$$

Equations (2.7), subject to appropriate boundary conditions, generally must be solved numerically because of their variable coefficients. It is the purpose of the present work to analyze the behavior of solutions to (2.7) in the vicinity of the throat of the duct when the throat Mach number $M(0)$ is close to unity. It is well known that the system (2.7) is singular at any point where $M(x)=1$. This can be seen most simply by subtracting the two equations; the resulting equation has no μ_x term, and the coefficient of r_x becomes $(M^2-1)/M$, which

vanishes as $M \to 1$. This can only occur at x=0 for the duct of Fig. 1. The singularity at x=0 implies that, in general, the acoustic quantities r and μ will be singular when the flow is sonic there. Thus, as we shall see in what follows, r and μ generally become arbitrarily large near x=0 as M(0) approaches unity, thereby violating the assumptions made in deriving (2.7) that μ, μ_x, r, and r_x all remain bounded.

As a result of the singular behavior of the system (2.7) for high subsonic Mach numbers in the throat region, linearized acoustic theory is inadequate to describe sound propagation in the duct; we must re-formulate the perturbation scheme to take into account nonlinear terms in the system (2.4) which were neglected in (2.7). However, in order to make progress in this direction it is necessary to know precisely the nature of the singular behavior of the solutions μ and r to (2.7). This behavior has been recognized, but never resolved, in previous treatments of the system (2.7) (refs. 4,5). In section 4 of this paper we construct an analytical general solution for the linear system (2.7) which displays explicitly the nature of its solutions at x=0 when M(0) is near unity. Before constructing this solution, however, we must discuss the behavior as $M(0) \to 1$ of the solutions to the steady flow equations (2.2) in some detail in the following section.

3. BASIC STEADY FLOW

As we have seen, the acoustic equations of motion are singular at x=0 when M(0)=1. It is useful, therefore, to introduce a parameter $\varepsilon = 1 - M(0)$ into our discussion and to consider both the basic steady flow quantities and the acoustic quantities as functions of ε; i.e., $U(x)=U(x;\varepsilon)$, $\mu(x,t)=\mu(x,t;\varepsilon)$, and so on. The parameter ε can be considered as having been introduced through the unstated boundary conditions on the steady flow.

The elementary equations of quasi-one dimensional flow (2.2) are discussed in detail in numerous texts; for example, a particularly comprehensive treatment is given by Crocco (ref. 8). It is straightforward to express any of the fluid quantities in terms of the duct area A(x) or, equivalently, in terms of the Mach number $M(x;\varepsilon)$. However, the behavior of M explicitly as a function of x and ε does not, to our knowledge, appear in the literature, and it is the purpose of the present section to determine this.

We begin with the well-known relation implied by equation (2.2),

$$M' = -MGA'/(1-M^2)A \tag{3.1}$$

which becomes, after integration,

$$\alpha^s M^s(x)[1+(\gamma-1)M^2(0)/2] = M^s(0)[1+(\gamma-1)M^2(x)/2] \tag{3.2}$$

where, in equation (3.2) we have defined,

$$A(x)/A(0) = \alpha(x), \quad s = 2(\gamma-1)/(\gamma+1) \tag{3.3}$$

Figure 2 shows a sketch of typical integral curves of equation (3.1) assuming $A''(0)\neq 0$. We are interested in a curve such as AB in figure 2, for which M remains less than unity for all x. Since $1-M(0)=\varepsilon$ is assumed small it is natural to seek an expansion of $M(x;\varepsilon)$ in the form

$$M(x;\varepsilon) = M_0(x) + \varepsilon M_1(x) + \varepsilon^2 M_2(x) \ldots \tag{3.4}$$

where $M_0(0)=1$. Substituting (3.4) into (3.2) and equating like powers of ε we find that $M_0(x)$ must satisfy

$$(\gamma+1)\alpha^s M_0^s /2 = 1+(\gamma+1)M_0^2/2 \tag{3.5}$$

while $M_1(x)=0$, and

$$M_2(x) = -\ 2M_0[1+(\gamma-1)M_0^2/2]/(\gamma+1)(1-M_0^2) \tag{3.6}$$

Obviously, as can also be inferred from the integral curves of figure 2, the expansion (3.4) is not uniformly valid near x=0: the third term is as large as the first whenever $1-M_0(x)$ is as small as ε.

It remains to find $M_0(x)=M(x;0)$ in terms of x; i.e., to solve equation (3.5). We express $\alpha(x)$ as a power series

$$\alpha(x) = 1 + ax^2 + \ldots$$

where we assume $a=A''(0)/2A(0)\neq 0$. Then $M_0(x)$ can be determined after some algebra in the form:

$$M_0(x) = 1 -((\gamma+1)a/2)^{\frac{1}{2}}|x| + \ldots \tag{3.7}$$

Thus, the leading term of $M(x;\varepsilon)$ behaves as a piecewise linear function of x near the throat so long as $a\neq 0$. If a=0 we find that $M_0(x)$ is smooth at x=0, but this case will not be discussed further in this paper.

4. SINGULARITY OF THE ACOUSTIC SOLUTION

We shall now analyze the acoustic equations (2.7) in order to exhibit explicitly the singular behavior of their solutions in the vicinity of the throat as M(0) approaches unity. Since the steady flow depends on the parameter $\varepsilon=1-M(0)$, the coefficients in the acoustic equations and hence the acoustic quantities μ and r are functions of ε. For $\varepsilon<<1$ we look for solutions of the acoustic equations in the form

$$r=r_0(x,t)+\varepsilon r_1(x,t)+ \ldots\ , \quad \mu=\mu_0(x,t)+\varepsilon\mu_1(x,t)+ \ldots \tag{4.1}$$

Inserting equation (4.1) into equation (2.7) and using expansion (3.4) for the coefficients, we get, after neglecting higher order terms,

$$G_0^{\frac{1}{2}} r_{0_t} + M_0(r_{0_x} + \mu_{0_x}) = 0$$
$$G_0^{\frac{1}{2}} \mu_{0_t} + M_0 \mu_{0_x} + (1/M_0) r_{0_x} + (M'_0/G_0)(2\mu_0 - (\gamma-1) r_0) = 0 \tag{4.2}$$

We recall that $M_0(x)$ is any Mach number distribution which yields a sonic velocity at the throat and a subsonic velocity throughout the remainder of the duct. These equations are singular at x=0 since $M_0(0)=1$ *. This is most directly seen by subtracting the two equations.

Analytical solutions of the system (4.2) cannot be found for arbitrary $M_0(x)$ and arbitrary time dependence. However for harmonic time dependence the system can be reduced to a system of ordinary differential equations with a singular point at x=0 and no other singular points within the duct. The nature of this singular point will determine the singular behavior in the time harmonic acoustic quantities. Explicit analytical results concerning the exact nature of the singular point and the dependence on $M_0(x)$ or $U_0(x)$ can be found by use of series solution methods for linear ordinary differential equations. We do not present the general results of this analysis in the current paper. Instead, we shall illustrate the singular behavior of a general solution of the system (4.2) corresponding to a specific steady flow. We assume that the time dependence is harmonic and that the steady velocity distribution is given by a piecewise linear function of x:

$$U_0(x) = c^*(1-K|x|) \qquad |x| < (1/K) \tag{4.3}$$

where c^* is the critical sound speed and K is a positive constant. This velocity distribution corresponds to a reasonably shaped duct with $A'(0)=0$ and $A''(0)\neq 0$. Equation (3.7) leads us to observe that for any duct with $A'(0)=0$ and $A''(0)\neq 0$, $U_0(x)$ will be given by equation (4.3) for x sufficiently close to the throat. Thus results of this section will be generally applicable to the throat region of many ducts of practical interest. The Mach number distribution associated with equation (4.3) is

$$M_0(x) = (1-K|x|)[(\gamma+1)/2 - (\gamma-1)(1-K|x|)^2/2]^{-\frac{1}{2}} \tag{4.4}$$

A general solution of the system (4.2) with $M_0(x)$ given by equation (4.4) can be constructed by judicious use of an analytical solution found by Crocco and Cheng (ref. 3). In effect, they obtained a general solution to the system (4.2) with

$$r_0(x,t) = r_0(x)\exp(i\Omega t) \ , \ \mu_0(x,t) = \mu_0(x)\exp(i\Omega t) \tag{4.5}$$

where $\Omega = \beta K(2/\gamma+1)^{\frac{1}{2}}$ and

$$U_0(x) = c^*(1+Kx) \ , \ -(1/K)<x \tag{4.6}$$

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*They are also singular since M_0' has a jump discontinuity at x=0. This does not affect the dominant singular behavior in the solution and would not be present if we had chosen $A''(0)=0$.

Their treatment involves introduction of the new independent variable $z=(1+Kx)^2$. Inserting equations (4.5) and (4.6) into the system (4.2) and introducing z and $\sigma(z)=r_0(x(z))$, $\nu(z)=\mu_0(x(z))$ we obtain, after elimination of $\nu(z)$,

$$z(1-z)d^2\sigma/dz^2-2[1+i\beta/(\gamma+1)]z\ d\sigma/dz - [i\beta(2+i\beta)/2(\gamma+1)]\sigma=0 \tag{4.7}$$

and

$$\nu(z) = [(\gamma-1+i\beta)\sigma-(\gamma+1)(1-z)\ d\sigma/dz]/(2+i\beta) \tag{4.8}$$

Equation (4.7) is a hypergeometric equation with complex coefficients. Two linearly independent solutions are

$$\sigma_1=F(a,b,d;1-z) \quad \text{and} \quad \sigma_2=(1-z)^{1-d}F(-b,-a,2-d;1-z) \tag{4.9}$$

where

$$d = 2 + 2i\beta/(\gamma+1)\ , \quad a+b+1 = d\ , \quad ab = i\beta(2+i\beta)/2(\gamma+1) \tag{4.10}$$

and F is the standard hypergeometric function (ref. 9).

Since the velocity distribution used by Cheng and that in equation (4.3) are identical for $x<0$, equation (4.9) provides a general solution to our problem for $x<0$. Of course, Cheng's solution for $x>0$ corresponds to a supersonic steady flow and is not relevant to our discussion. In order to obtain a solution when $U_0(x)$ is given by equation (4.3) for $x>0$ we observe that the acoustic equations in this case reduce to equations (4.7) and (4.8) if β is replaced by $-\beta$. Thus we have found a general solution to the acoustic equation corresponding to $U_0(x)$ given by equation (4.3) for both $x<0$ and $x>0$.

The singularity at $x=0$ can be found explicitly by examining the solutions σ_1 and σ_2 of equation (4.9) and the corresponding functions when β is replaced by $-\beta$. Clearly σ_1 is analytic at $z=1(x=0)$ and the singular behavior is due to σ_2. The leading term in σ_2 for z near unity is given by

$$\sigma_2\sim(1-z)^{1-d}F(a,b,d;0) = [\cos(q\ln(1-z))\mp i\ \sin(q\ln(1-z))]/(1-z) \tag{4.11}$$

where $q=2\beta/(\gamma+1)$ and the - and + signs hold for $x<0$ and $x>0$, respectively (ref. 9). For general acoustic boundary conditions both σ_1 and σ_2 appear in the acoustic solution, and the amplitudes of the acoustic quantities will approach infinity as x^{-1} when $x\to0$. In addition their phases have an oscillatory discontinuity at $x=0$. Figure 3 shows an example typical of the behavior of the acoustic quantities (pressure, in this case) for small ε. The rapid rise in the vicinity of the throat is indicative of the developing singularity in the linearized acoustic quantities as $\varepsilon\to0$. For the typical case shown a pressure wave of magnitude unity was incident from the left on a converging-diverging section situated in an otherwise uniform duct.

Since any duct with a tnroat at which $A''(0)\neq0$ will have a locally (near $x=0$) piecewise linear steady velocity distribution, the acoustic quantities in such a duct will have the singular behavior given in equation (4.11). In this

circumstance the linear theory governing expansion (2.6) fails for any nonzero δ, no matter how small. Nonlinear effects become appreciable in the throat region. In the next section, an approach is outlined which applies the method of matched asymptotic expansions to study the nonlinear effect in the region near the throat.

5. NONLINEAR PERTURBATION EQUATIONS

In this section we set forth in summary the theory describing sound propagation near the throat as $M(0)\to 1$. We regard the expansion indicated by equations (2.3) and (2.6) as an outer expansion valid as $\delta\to 0$ for fixed values of x, t, and ε; i.e.,

$$\begin{aligned}\bar{u}(x,t;\varepsilon,\delta) &= U(x;\varepsilon)[1 + \delta\mu(x,t;\varepsilon) + \dots]\\ \bar{\rho}(x,t;\varepsilon,\delta) &= R(x;\varepsilon)[1 + \delta r(x,t;\varepsilon) + \dots]\end{aligned} \tag{5.1}$$

From the details of the previous section we know that in general μ and r become arbitrarily large in the limit as x and ε approach zero, being expected to grow as x^{-1}.

Therefore we introduce an inner variable $X=[(\gamma+1)/2]^{\frac{1}{2}}(x/\varepsilon)$ and assume that $\bar{u}$ and $\bar{\rho}$ behave asymptotically as $\varepsilon\to 0$ with X, t fixed as:

$$\begin{aligned}\bar{u} &= \bar{u}^{i}(X,t;\varepsilon) = U^{i}(X;\varepsilon)[1+\varepsilon\mu^{i}(X,t) + \dots]\\ \bar{\rho} &= \bar{\rho}^{i}(X,t;\varepsilon) = R^{i}(X;\varepsilon)[1+\varepsilon r^{i}(X,t) + \dots]\end{aligned} \tag{5.2}$$

where ε is assumed to be a function of δ which vanishes as $\delta\to 0$ and is to be determined by asymptotic matching of (5.2) with (5.1). In addition we expand the steady flow quantities in the form:

$$U^{i}(X;\varepsilon)=U_0^{i}(X)+\varepsilon U_1^{i}(X)+\dots,\quad R^{i}(X;\varepsilon)=R_0^{i}(X)+\varepsilon R_1^{i}(X)+\dots \tag{5.3}$$

Equations (5.3) could be combined directly with (5.2) as one inner expansion, but we find that it simplifies the considerable algebra involved to retain the dimensionless perturbations μ and r and to multiply the separate expansions as indicated in equation (5.2).

The steady flow quantities in equation (5.3) are known if the corresponding expansion of the Mach number is known. We assume an expansion for M of the same form as (5.3), substitute this expansion into equation (3.1), transform to the inner variable X, and solve the successive differential equations which result. After matching with the outer expansion (3.4) we find

$$M(x;\varepsilon) = M^{i}(x;\varepsilon) = 1 - \varepsilon\ (1+aX^2)^{\frac{1}{2}} + \dots \tag{5.4}$$

where $a=A''(0)/2A(0)$ as before. Using (5.4) we find the coefficients in the expansions (5.3) by use of the steady flow relations between M and U or R.

Next we carry out the same process on system (2.4) using $u=\varepsilon\mu^i+ \ldots$, $\rho=\varepsilon r^i+\ldots$, and substituting equation (5.4) in the coefficients. This yields the system of inner equations satisfied by μ^i and r^i in the form:

$$\begin{aligned} &\eta_t + [\zeta+\eta-(1+aX^2)^{\frac{1}{2}}]\,\eta_X - \frac{aX}{(1+aX^2)^{\frac{1}{2}}}(\zeta+\eta) = 0 \\ &\zeta_X = 0 \end{aligned} \tag{5.5}$$

where in equations (5.5) we have defined

$$\zeta = (3-\gamma)(\mu^i+r^i)/4 \quad \text{and} \quad \eta = (\gamma+1)(\mu^i-r^i)/4$$

Equations (5.5) are the nonlinear equations which, to first order in ε, govern sound propagation through a throat as the Mach number there approaches unity. The lengthy details of solving the system will not be presented here. However, certain important conclusions can be made at this stage. The quantities η and ζ are related to the Riemann invariants of system (2.4), η representing the upstream and ζ the downstream propagating portions of the solution to (2.4). Considerations of asymptotic matching between expansions (5.1) and (5.2) lead to the conclusion that, to first order in ε, ζ actually vanishes. Thus, as is often argued from physical considerations, the lowest order nonlinear effect of the sonic throat is on the upstream propagating waves alone.

A final observation which we make here is that matching considerations indicate that, in the distinguished limit implied by the inner expansion (5.2), ε is to be taken equal to $\delta^{\frac{1}{2}}$. Hence we conclude that, given an acoustic source strength δ, nonlinear effects on sound arise for throat Mach numbers as far away from unity as $\delta^{\frac{1}{2}}$. This would explain why marked sonic throat effects are observed experimentally for throat Mach numbers as low as 0.75-0.8.

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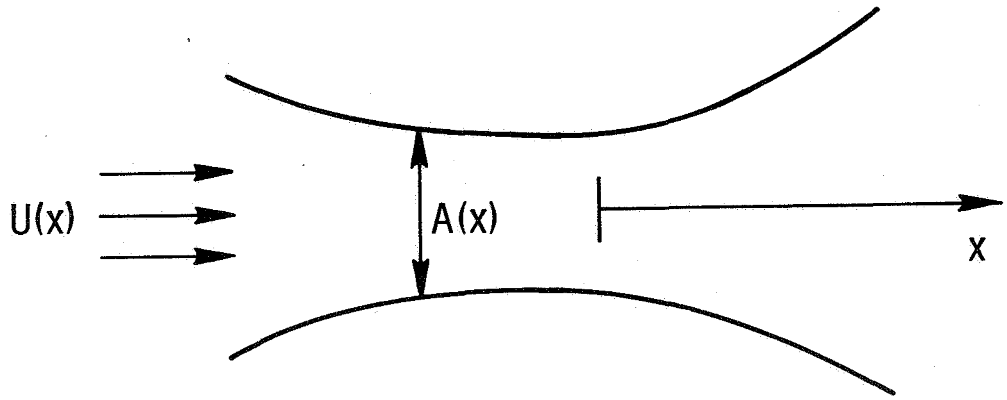

Figure 1.- Sketch of typical duct geometry.

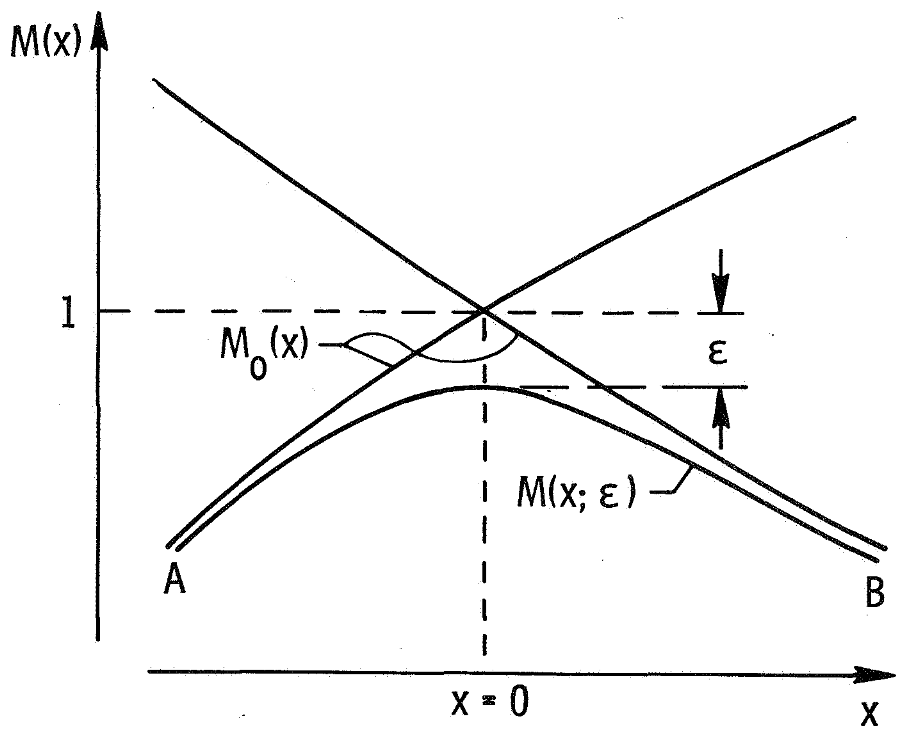

Figure 2.- Typical steady flow integral curves. Curve AB is described by equations (3.4) and (5.4).

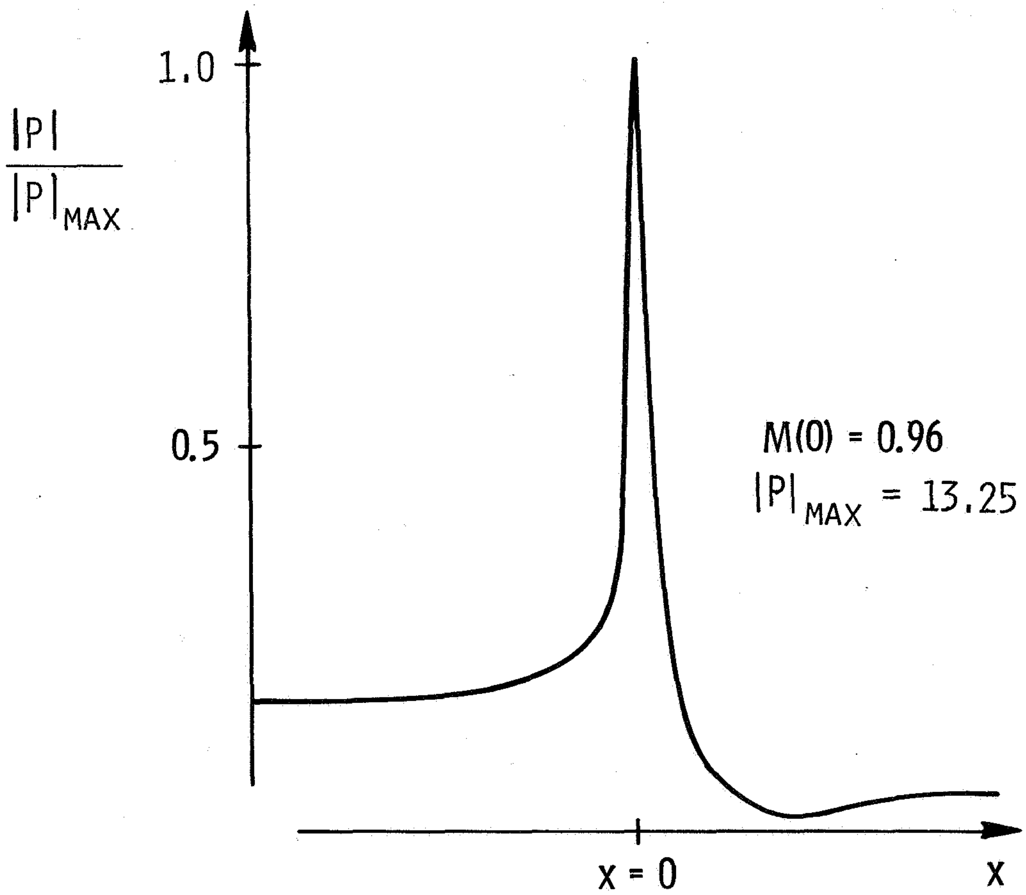

Figure 3.- Typical behavior of linearized pressure magnitude for high subsonic throat Mach numbers.